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Uncertainty about fundamental and pessimistic traders: a piecewise-linear maps approach

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Abstract

We analyze a financial market model with heterogeneous interacting agents where two groups of fundamentalists are taken into account. We assume that agents are homogeneous in their trading strategy, but heterogeneous in their belief about the fundamental value of the asset. Moreover, agents are characterized by an underlying pessimism due to the uncertainty about the fundamental value. As a consequence, the dynamics of our model is driven by a one-dimensional piecewise-linear continuous map with three linear branches. We investigate the bifurcation structures in the parameter space of the map and describe the endogenous fear and greed market dynamics arising from our asset-pricing model.

Keywords: Piecewise-linear maps; Border Collision Bifurcations; Fear and Greed; Market Risk

1 Introduction

In this paper we contribute to the financial literature in understanding of what is driving the dynamics of financial markets. In particular, we focus on the role of qualitative Dynamical Systems Theory in modelling financial markets populated by heterogeneous agents as described in the pioneering works of Day and Huang (1990), Brock and Hommes (1998). Several papers have been able to replicate the stylized facts of financial markets such as volatility clustering, asymmetry and excess of kurtosis documented in Lux and Marchesi (2000) and Cont (2001) among others. Most of these works rely on the introduction of a non-linearity component in the model, some examples are Bischi *et al.* (2006), Chiarella *et al.* (2006), He and Li (2007). As explained in Tramontana *et al.* (2010b), non-linearity is used to build the structure of several heterogeneous interacting agents in financial markets. In particular, there are three main basic frameworks: introducing non-linearity in the trading rules as in Day and Huang (1990) and Chiarella *et al.*

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(2002). Several contributions are based on the Adaptive Belief System outlined by Brock and Hommes (1998) where investors adapt their beliefs over time by choosing from different predictors of future values of endogenous variables, see e.g. Brianzoni *et al.* (2010), Franke and Westerhoff (2016), Dercole and Radi (2020). Finally, a third mechanism of endogenous dynamics is based on market interactions where connections are due to the trading activity of heterogeneous speculators. Follow this line of research the works of Chiarella *et al.* (2005), Tramontana *et al.* (2009) and Dieci and Westerhoff (2010).

A further framework on building agent-based financial market models proposes the approach of the piecewise-linear maps. Piecewise-linear (PWL) maps can be considered as an approximation of more complicated non-linear maps, and this embodies an advantage given that they allow us an extensively analytical study of the model. Moreover, PWL maps exhibit a further bifurcation structures with respect those occurring in the standard maps: the border collision bifurcations (see Nusse and Yorke (1992); Nusse *et al.* (1994)). Border collision bifurcations (BCB) are a peculiar phenomenon occurring only in non-smooth systems (continuous or discontinuous). These bifurcations are caused by the merging of some invariant set (i.e. fixed points, periodic cycles or boundary of any invariant set) with the kink point at which the function changes its definition (see Avrutin *et al.* (2019) and Gardini and Tramontana (2011)). Only few models are represented by this kind of maps, some related contributions are Day (1997), Tramontana *et al.* (2010b), Tramontana *et al.* (2014). Moreover, all the models considered involve fundamentalists and chartists (Tramontana *et al.* (2010b,a, 2014, 2015)) and imitators (Campisi and Tramontana (2020) and Brianzoni and Campisi (2020)). Unlike previous works in PWL maps set-up, we build a financial market populated with only fundamentalists following the contribute of Naimzada and Ricchiuti (2008), De Grauwe and Kaltwasser (2012) and Campisi and Muzzioli (2020) in the contest of non-linear smooth dynamical systems. In particular, we assume that traders are homogeneous in their trading strategy, but heterogeneous in their beliefs about the fundamental value of the asset. This assumption introduce a degree of uncertainty in the market as outlined in He and Zheng (2016). Fundamentalists take into consideration the difference between their perceived fundamental price and the market price. The resulting map is piecewise-linear and continuous with three linear branches showing interesting Border Collision Bifurcations phenomena. Moreover, we consider agents sharing a pessimistic mood, quite common in this of uncertainty due to the Covid-19 health emergency.

The contribution of the paper is as follows. First, we develop a financial market modelled as a one-dimensional continuous piecewise-linear map with three linear branches where two types of fundamentalists interact. Our model may be regarded as a PWL version of the model of Campisi and Muzzioli (2020). Second, we concentrate on the mathematical properties of the model providing a comprehensive analysis of the model dynamics. Depending on the slopes of the linear branches, we are able to describe three different cases: case I where we consider that all three branches are increasing and the map can admit convergence to an equilibrium or divergence from it. Cases II and III where the map has at least one decreasing branch, then beyond the possible paths seen in the previous case, now also periodic and chaotic dynamics may emerge. The three linear branches of our map correspond to three market scenarios: fear scenario (period of declining prices), greed scenario (periods of rising prices), fear and greed scenario. We have chosen this terminology instead of the usual bear and bull market to stress the role of uncertainty in the market we introduce about the fundamental, and to be closely related to the terminology of market risk (see for example Elyasiani *et al.* (2020) and Westerhoff (2004)). Finally, as remarked in Bischi *et al.* (2010a), Sushko *et al.* (2015) and Avrutin *et al.* (2019), a feature of piecewise-linear maps with two kinks is that they may exhibit coexisting attractors for opportune values of the parameters. For this purpose, we conduct numerical analysis to better investigate the multistability property of our model.

The rest of the paper is organized as follows. Section 2 reviews the related literature and highlights the contribution of the present paper. In Section 3 we introduce our financial market model. In Section 4 we conduct an in-depth analytical study of the model. In particular, we find the fixed points and examine their local and global stability. For this purpose, we also carry out a numerical analysis in order to support the analytical results. Section 5 concludes.

2 Literature review

In this section we give an overview of the most significant works analyzing the dynamics of a financial market by means of PWL maps, moreover we also highlight the main contributions of our study.

The contributions of PWL maps to the understanding of financial markets are very recent, with a few exceptions (see Day (1997) and Huang and Day (1993)). The relevant literature on PWL financial models considers mainly one-dimensional models due to the complexity and limitations to face models in higher dimensions. In this direction go the works of Panchuk *et al.* (2018) Tramontana and Westerhoff (2016),Tramontana *et al.* (2015),Tramontana *et al.* (2014), Tramontana *et al.* (2013), Tramontana and Westerhoff (2013), Tramontana *et al.* (2010b), Tramontana *et al.* (2010a). All these contributions may be considered extensions of the models studied in the seminal papers of Day and Huang (1990) and Huang and Day (1993). For example, Tramontana *et al.* (2014) study an asset price model with chartists and fundamentalists resulting in a one-dimensional map with three linear branches and two discontinuity points. A feature of their model relies on the fact that only two or three branches of the map are involved in the dynamics, moreover, in the latter case they show that intricate bull and bear market dynamics may emerge. Tramontana *et al.* (2015) extend the model of Tramontana *et al.* (2014) finding a particular scenario characterized by the existence of periodic and quasi-periodic dynamic behaviors.

Other studies focus on asset price dynamics using two-dimensional PWL maps, this is the case of Brianzoni and Campisi (2020). Gu and Guo (2019), Gu (2018). In particular, the recent work of Anufriev *et al.* (2020) analyzes an asset pricing model with three types of traders. Their model is characterized by an asymmetric propensity to trade between bull and bear markets. The multi-stability features underlined by the authors are able to describe a rich set of financial scenarios. Moreover, in their stochastic version of the model several empirical stylized fact are observed. Recently, Jungeilges *et al.* (2021) generalize the model of Day and Huang (1990). In particular they focus only on the asymmetry in no-trade intervals resulting in a discrete-time piecewise-linear model with 5-pieces linear branches.

Our study is related to the strand of literature examining financial markets with heterogeneous agents using one-dimensional PWL maps. Unlike previous works outlined, we assume that traders are homogeneous in their trading strategy but heterogeneous in their beliefs about the fundamental value of the asset. In this line of reasoning, we are introducing uncertainty in the market about the fundamental value following the indirect approach of Campisi and Muzzioli (2020) and Naimzada and Ricchiuti (2008) as opposed to He and Zheng (2016) where the uncertainty is directly modelled. However, all these works rely on the framework of the smooth non-linear dynamical systems, while our model is built up as a one-dimensional piecewise-linear map with three linear branches. To the best of our knowledge this is the first paper that analyze a one-dimensional PWL map with two fundamentalists. Moreover, we are able to show that depending on the slope of the branches of the map, the resulting dynamics consists of convergence to an equilibrium or divergence from it (when all the branches are increasing) or in the most interesting case, periodic and chaotic price dynamics (when at least one branch is decreasing). Although the model is deterministic, it replicates several aspects of stock market fluctuations quite well, such as bubbles and crashes. Indeed, one of the characteristics of PWL maps is that they may exhibit a sharp transition to chaotic dynamics, as we will see also in our analysis. This important feature of PWL maps resembles the sudden crisis analyzed in Huang *et al.* (2010).

3 The model

Our model incorporates two types of fundamentalists: in particular, we assume that type-2 fundamentalists (f_2) underestimate the fundamental value with respect to type-1 fundamentalists (f_1), i.e. $F_2 < F_1$. This implies that when the price (P_t) is lower than F_2, type-2 fundamentalists think the price is underestimated less than type-1 fundamentalists. On the other hand, when price P_t is higher than F_1, type-2 fundamentalists think the price P_t is overestimated more than type-1 fundamentalists. If the price P_t is higher than F_2 but lower than F_1 then type-1 fundamentalists think that the price is underestimated while type-2 fundamentalists think at an overestimation and they behave in an opposite way. We assume that both types of fundamentalist have the same excess demand function:

$$D_t^{f_1} = f_1(F_1 - P_t) - \mu_1 \tag{1}$$

$$D_t^{f_2} = f_2(F_2 - P_t) - \mu_2 \tag{2}$$

where $D_t^{f_i}$ is the excess demand function for type-i fundamentalists; $i = 1, 2$, f_i is a positive parameter and indicates how aggressively the fundamentalist reacts to the distance of the price to the corresponding fundamental value (F_1, F_2); $i = 1, 2$. The positive parameters μ_1, μ_2 capture some general kind of pessimism of investors.

Depending on the price, we can distinguish the following fear or greed predominance regions:

a) $P_t < F_2 < F_1$, both types of fundamentalists become less pessimist and sell a less amount or buy, but type-2 fundamentalists are more pessimist than type-1 fundamentalists (greed predominance region).

b) $F_2 < P_t < F_1$, type-2 fundamentalists sell, whereas type-1 fundamentalists sell a less amount or buy. This is similar to the bull and bear regime described in Tramontana *et al.* (2013) (fear and greed mixed predominance region).

c) $F_1 < F_2 < P_t$, in this case both fundamentalists sell, type-2 fundamentalists sell more than type-1 (fear predominance region).

In this context we can rewrite the demand function of both fundamentalists as follow:

$$D_t^{f_1} = \begin{cases} f_1(F_1 - P_t) - \mu_1 & \text{if} \quad P_t < F_2 < F_1 \\ f_1(F_1 - P_t) - \mu_1 & \text{if} \quad F_2 < P_t < F_1 \\ -f_1(F_1 - P_t) - \mu_1 & \text{if} \quad F_1 < F_2 < P_t \end{cases} \tag{3}$$

$$D_t^{f_2} = \begin{cases} f_2(F_2 - P_t) - \mu_2 & \text{if} \quad P_t < F_2 < F_1 \\ -f_2(F_2 - P_t) - \mu_2 & \text{if} \quad F_2 < P_t < F_1 \\ -f_2(F_2 - P_t) - \mu_2 & \text{if} \quad F_1 < F_2 < P_t \end{cases} \tag{4}$$

The market maker adjusts the price at time $(t+1)$ following this rule:

$$P_{t+1} = P_t + a(D_t^{f_1} + D_t^{f_2}) \tag{5}$$

The final map is:

$$P_{t+1} = T(P_t) = \begin{cases} f_L(P_t) = P_t + f_1(F_1 - P_t) + f_2(F_2 - P_t) - m & \text{if} \quad P_t < F_2 < F_1 \\ f_M(P_t) = P_t + f_1(F_1 - P_t) - f_2(F_2 - P_t) - m & \text{if} \quad F_2 < P_t < F_1 \\ f_R(P_t) = P_t - f_1(F_1 - P_t) - f_2(F_2 - P_t) - m & \text{if} \quad F_1 < F_2 < P_t \end{cases} \tag{6}$$

where $m = \mu_1 + \mu_2$. This is a continuous piecewise-linear map with two kink points at $P_t = F_1$ and $P_t = F_2$. In order to better study it we rewrite the map with the equations of the three linear branches in explicit form:

$$P_{t+1} = T(P_t) = \begin{cases} f_L(P_t) = (1 - f_1 - f_2)P_t + f_1F_1 + f_2F_2 - m & \text{if} \quad P_t < F_2 < F_1 \\ f_M(P_t) = (1 - f_1 + f_2)P_t + f_1F_1 - f_2F_2 - m & \text{if} \quad F_2 < P_t < F_1 \\ f_R(P_t) = (1 + f_1 + f_2)P_t - f_1F_1 - f_2F_2 - m & \text{if} \quad F_1 < F_2 < P_t \end{cases} \tag{7}$$

4 Study of the map

In this Section we study analytically and numerically the main properties of the map (7). First, we make some general considerations about the particular kind of piecewise-linear map we dealt with in this paper, and then we focus on the Border Collision Bifurcations that may originate by varying the model's parameters.

4.1 General Features

Map (7) is made up by three linear pieces, connected at the two kink points. From now on we refer to the three branches as L(eft), M(iddle) and R(ight) branch, respectively.

The slope of the right branch ($s_R = 1+f_1+f_2$) is larger than the slope of the middle one ($s_M = 1-f_1+f_2$), which is larger than the slope of the left branch ($s_L = 1 - f_1 - f_2$). This is a consequence of the positive values that the reactivity parameters may assume.

In particular, the left branch can be both positive and negative, but the slope cannot be higher than one. The middle branch can also be both increasing and decreasing and the slope may take any value. The right slope can only be increasing, with slope higher than one.

So, according to the (positive) values of f_1 and f_2 we may have three possible scenarios:

Case I Increasing/Increasing/Increasing (if $f_1 < 1 - f_2$)

Case II Decreasing/Increasing/Increasing (if $1 - f_2 < f_1 < 1 + f_2$)

Case III Decreasing/Decreasing/Increasing (if $f_1 > 1 + f_2$)

This is sufficient to state that the map may admit up the two equilibria. Three coexisting equilibria are not possible. In fact, in Cases I and II, if the increasing middle branch intersects the bisector creating an equilibrium, the right branch, increasing with higher slope, cannot intersect the bisector. In Case 3 the left and middle branches, both decreasing, cannot both intersect the bisector. This intuitive result can be confirmed analytically as we do in the following Proposition.

Proposition 1 *Map (7) may admit either two equilibria or no equilibria at all. It admits no equilibria when:*

$$\frac{m}{F_1 - F_2} < f_i \quad \text{for } i = 1, 2 \tag{8}$$

Otherwise, it admits two equilibria, one on the M and the other on the R branch when:

$$f_2 < \frac{m}{F_1 - F_2} < f_1, \tag{9}$$

two equilibria, one on the L and the other on the M branch when:

$$f_1 < \frac{m}{F_1 - F_2} < f_2, \tag{10}$$

two equilibria, one on the L and the other on the R branch, when:

$$f_i < \frac{m}{F_1 - F_2} \quad \text{for } i = 1, 2 \tag{11}$$

Proof 1 *The left branch intersects the bisector at:*

$$\overline{P}_L = \frac{f_1 F_1 + f_2 F_2 - m}{f_1 + f_2} \tag{12}$$

which is an equilibrium of the map if it is lower than F_2, that is iff:

$$f_1 < \frac{m}{F_1 - F_2} \tag{13}$$

The middle branch intersects the bisector at:

$$\overline{P}_M = \frac{f_2F_2 - f_1F_1 + m}{f_2 - f_1} \tag{14}$$

which is an equilibrium of the map if it is higher than F_2 and lower than F_1, that is iff:

$$f_1 < \frac{m}{F_1 - F_2} < f_2 \tag{15}$$

or

$$f_2 < \frac{m}{F_1 - F_2} < f_1 \tag{16}$$

The right branch intersects the bisector at:

$$\overline{P}_R = \frac{m + f_1F_1 + f_2F_2}{f_1 + f_2} \tag{17}$$

which is an equilibrium of the map if it is larger than F_1, that is iff:

$$f_2 < \frac{m}{F_1 - F_2} \tag{18}$$

By putting together the existence conditions of the equilibria we obtained what is stated in the Proposition.
■

Proposition 1 implies that it is not possible to have only one equilibrium or three coexisting equilibria. More in detail, two equilibria born via fold-BCB and one of them is located in the middle branch. Then the equilibrium of the middle branch, by moving a parameter, may move to the third branch.

As an example, let us consider the panels of Figure 1. In all the panels we keep fixed the parameters' values: $f_1 = 0.9$, $f_2 = 0.15$, $F_1 = 1.5$ and $F_2 = 1$. In panel (a) we have $m = 0$, so $\frac{m}{F_1 - F_2} = 0 < f_2 < f_1$ and there are no equilibria. In panel (b) we have $m = 0.075$ and $\frac{m}{F_1 - F_2} = 0.15 = f_1 < f_2$ and two coincident equilibria are created via fold-BCB at $P = F_1$ (that is in the kink between the middle and the right branch. In panel (c) we have $m = 0.22$ and $f_1 < \frac{m}{F_1 - F_2} = 0.44 < f_2$ and we have two distinct equilibria in the middle and in the right branch. In panel (d) we have $m = 0.45$ and $f_1 < \frac{m}{F_1 - F_2} = 0.9 = f_2$ and the equilibrium of the middle branch merges with the kink at $P = F_1$. Finally, in panel (e) we have $m = 0.86$ and $f_1 < f_2 < \frac{m}{F_1 - F_2} = 1.72$ and the equilibrium from middle branch is now on the left one, while the second equilibrium is still on the right branch.

4.2 Stability of the equilibria

In order to understand the outcome of the price dynamics, the study of the existence of the equilibria is not sufficient without the study of their (local) stability.

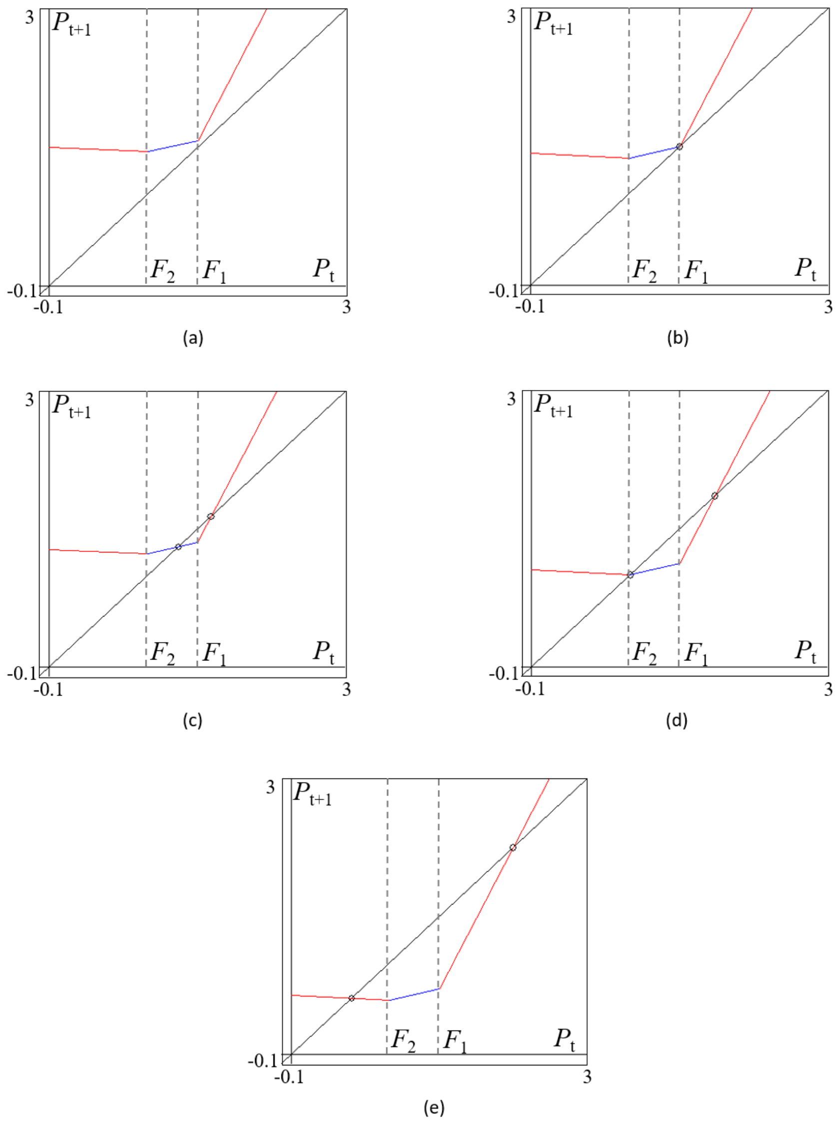

Figure 1: Parameter values: $f_1 = 0.9$, $f_2 = 0.15$, $F_1 = 1.5$ and $F_2 = 1$. In panel (a) we have $m = 0$. In panel (b) $m = 0.075$. In panel (c) $m = 0.22$. In panel (d) $m = 0.45$, while in panel (e) $m = 0.86$.

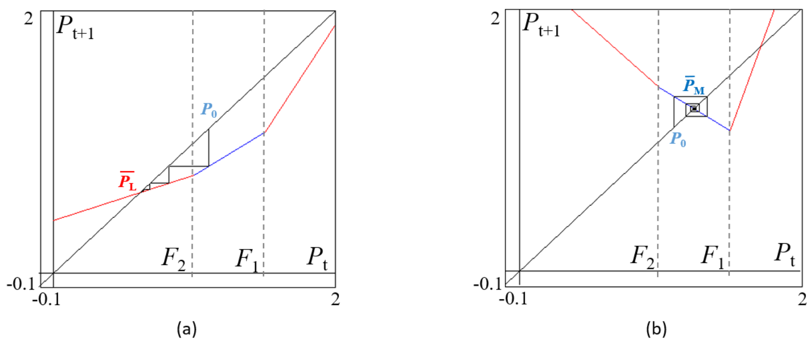

Figure 2: In panel (a) parameter values $m = 0.6,\ f_1 = 0.6,\ f_2 = 0.15,\ F_1 = 1.5,\ F_2 = 1$. In panel (b) $m = 0.5,\ f_1 = 1.8,\ f_2 = 0.1,\ F_1 = 1.5,\ F_2 = 1$.

Considering that our map is piecewise-linear this means that in order to have a stable equilibrium, existence conditions must be combined with the condition that the slope of the branch where the equilibrium is located should be lower than one in absolute value. This permits to obtain the following result:

Proposition 2 *Map (7) may admit only one locally stable equilibrium, the one located on the left branch or the one located on the middle branch.*

Proof 2 *The Equilibrium located on the left branch ($\overline{P}_L$) exists and is locally stable iff:*

$$f_1 < \frac{m}{F_1 - F_2} \ \cup \ f_1 < 2 - f_2 \tag{19}$$

while the one located on the middle branch ($\overline{P}_M$) exists and is locally stable iff:

$$f_2 < \frac{m}{F_1 - F_2} < f_1 \cup \ f_2 < f_1 < 2 + f_2 \tag{20}$$

Considering that f_1 cannot be at the same time larger and lower than $\frac{m}{F_1 - F_2}$, this proves that they cannot be both locally stable.

Moreover, considering that the slope of the right branch is always larger than one, the equilibrium located on the right branch ($\overline{P}_R$), when it exists, is always unstable. ■

In Figure 2 we have an example of converging trajectory towards $\overline{P}_L$ (panel (a)) and converging trajectory towards $\overline{P}_M$ (panel (b)).

In order to better investigate the kind of dynamics that may occur and the attractors involved, we start by considering the scenario I and then we will consider the other two scenarios.

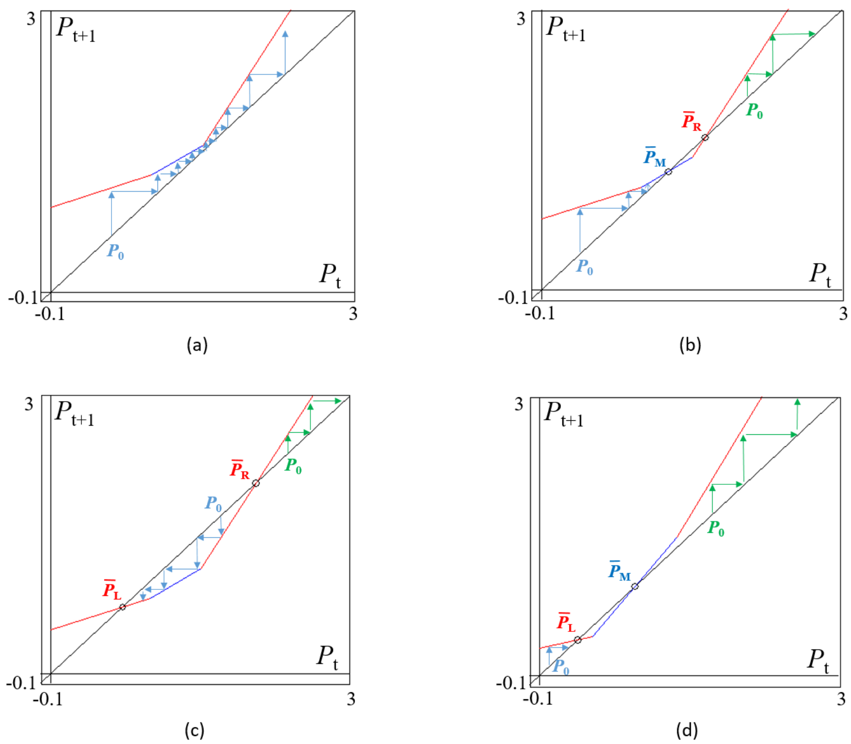

Figure 3: Parameter values: $m = 0.5$, $F_1 = 1.5$, $F_2 = 1$, $f_2 = 0.15$. In panel (a) we have that $f_1 = 0.5$, while in panel (b) $f_1 = 1.8$.

4.3 Dynamics in Scenario I (Incr/Incr/Incr)

As we know this scenario occurs when:

$$f_1 < 1 - f_2 \tag{21}$$

In this case when there are no equilibria, the three branches are all over the bisector and only diverging trajectories may exist (Figure 3, panel (a)).

At the opposite, when there are two coexisting equilibria, if one of them is located on the right branch, then the other one (which can be either on the left or in the middle branch) is locally stable. Initial conditions $P_0 > \overline{P}_R$ lead to divergence, while with $P_0 < \overline{P}_R$ trajectories converge to the stable equilibrium (Figure 3, panel (b) and (c)).

If the two coexisting equilibria are located one the left and on the middle branch, then $\overline{P}_M$ is necessarily unstable, while $\overline{P}_L$ is locally stable. Initial conditions $P_0 < \overline{P}_M$ lead to convergence to $\overline{P}_L$, otherwise we have divergence (Figure 3, panel (d)).

So in this scenario dynamics either converge to an equilibrium or diverge and in case of coexistence of equilibria the initial condition is fundamental to have one kind of dynamics or the other.

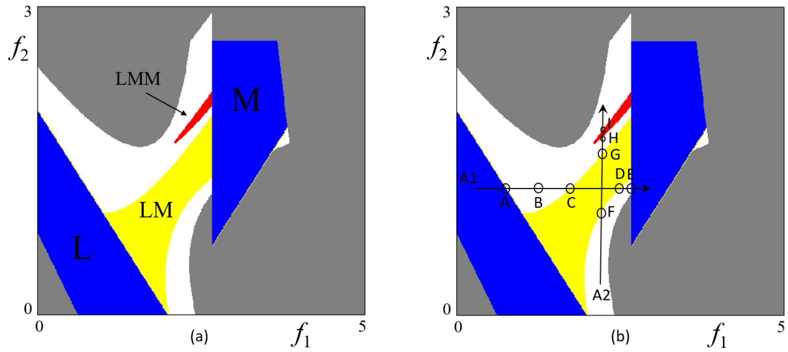

Figure 4: In blue combinations of parameters leading to an equilibrium, in yellow combinations leading to a cycle of period 2, in red to a cycle of period 3. In white chaotic dynamics, while in grey diverging trajectories.

4.4 Dynamics in Scenarios II and III

The other two scenarios, with at least one decreasing branch, may originate more interesting dynamics. In particular, not only convergence to an equilibrium or divergence are possible, but also periodic and chaotic dynamics. Let us consider the two-dimensional bifurcation diagram of Figure 4a, obtained by keeping fixed the parameters $F_1 = 1.3$, $F_2 = 1$, $m = 0.8$ and by letting vary f_1 and f_2. As we can see, besides the blue regions denoting convergence to an equilibrium and the grey one denoting divergence, there is a yellow region where dynamics converge to a cycle of period 2 (with a point on the left branch and one in the middle) and a red region corresponding to a cycle of period 3 (with one point on the left branch and the other two on the middle one). Moreover, the white region denotes chaotic trajectories.

If we focus on the directions of the arrows A1 and A2 we obtain the one-dimensional bifurcation diagrams shown in Figure 4b.

Let us start by considering the bifurcations occurring moving along arrow A1 of Figure 4b. In point A we have the sudden transition from convergence to the equilibrium $\overline{P}_L$ on the left branch, to chaos. Figure 6a shows that in that case the equilibrium becomes unstable via degenerate flip (the slope of the left branch is equal to -1). In the (f_1, f_2) parameter plane, the equation of the degenerate flip bifurcation curve of the equilibrium $\overline{P}_L$, that we denote with Ψ_L, is the following:

$$\Psi_L : f_2 = 2 - f_1 \tag{22}$$

After that, dynamics become chaotic. In particular, the bifurcation diagram of Figure 5a permits to conclude that the occurring chaotic attractor is made up by two bands. Figure 6b, representing point B of Figure 4b, is an example of it. By keep increasing the value of f_1 we reach point C, where from chaos we move to an attracting cycle of period 2, with one point on the left branch and the second one in the middle branch. The cycle originated through degenerate flip, in fact, in C the product of the slopes of the left and middle branch is equal to -1 (see Figure 6c). The equation of the degenerate flip bifurcation curve of the 2-cycle, that we denote with Ψ_{LM}, is the following:

$$\Psi_{LM} : f_2 = \sqrt{f_1^2 - 2f_1 + 2} \tag{23}$$

The cycle of period 2 undergoes a BCB for a value of f_1 denoted by point D in Figure 4b. Here, a point of the 2-cycle collides with the border F_1(see Figure 6d). The BCB curve of the 2-cycle (that we denote with Φ_{LM}) cannot be obtained analytically, but we can obtain it numerically. After the BCB of the 2-cycle we have again chaos (in a unique band, see the bifurcation diagram in Figure 5a) until we reach point E, where the equilibrium in of the middle branch ($\overline{P}_M$) undergoes a BCB with the border F_2 [1] (see Figure 6e). The equation of the BCB of the equilibrium $\overline{P}_M$ (that we denote with Φ_{M2}) is the following:

$$\Phi_{M2} : f_1 = \frac{m}{F_1 - F_2} \tag{24}$$

After that, in the final part of arrow A1, we have convergence to $\overline{P}_M$.

Let us consider now the arrow A2 of Figure 4b. In that case f_1 is fixed at a value of 2.31 and we increase the value of f_2.

In point F of Figure 4b we pass from chaos to a cycle of period 2, which is created via BCB. This point belongs to the same curve of point D, that is to the curve Φ_{LM}. When the value of f_2 is in point G, the cycle of period 2 undergoes a degenerate flip that we have already characterized (it is Ψ_{LM}, that contains also point C).

Then we have chaos until point H is reached. In that point a cycle of period 3 with one point on the left branch and two points on the middle branch is created via BCB. Figure 6f shows that the second point of the cycle in order of value collides with F_2. The cycle of period 3 becomes unstable in point I because of a degenerate-flip bifurcation of the cycle. The product of the slopes of the branches where the three points are located becomes -1. Then, we have chaos again.

As we can see from Figure 7ab the bifurcation curves that we have obtained permit to explain the two-dimensional bifurcation diagram of Figure 4. We have also drawn the BCB curve of the equilibrium in the middle branch ($\overline{P}_M$) when it collides with the border F_1 (denoted with Φ_{M1}) and the degenerate-flip bifurcation curve of the same equilibrium (denoted with Ψ_M), whose equations are:

$$\Phi_{M1} : f_2 = \frac{m}{F_1 - F_2} \tag{25}$$

$$\Psi_M : f_2 = f_1 - 2 \tag{26}$$

4.5 Multistability

A feature of a piecewise-linear map with two kinks is that it is possible to have values of the parameters at which two attractors coexist (Avrutin *et al.* (2019)). An example is shown in Figure 8a, where we have coexistence of a chaotic attractor made up by two bands and the locally stable equilibrium in the

[1] Actually we could also look at that as a BCB of the equilibrium $\overline{P}_L$ because in (e) $\overline{P}_L$ and $\overline{P}_M$ are coincident.

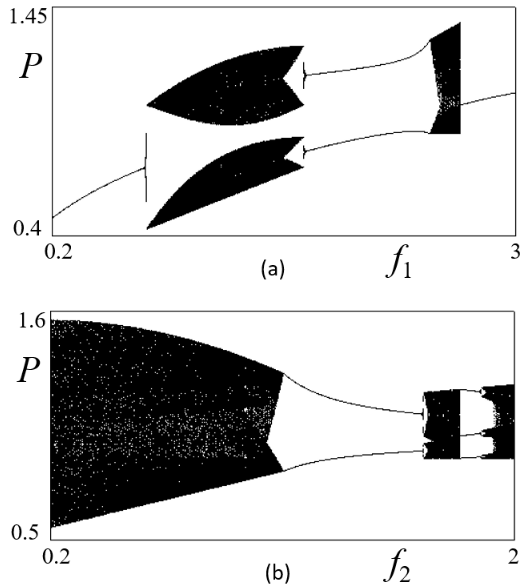

Figure 5: In both panels we used the same fixed parameters used for Figure 4. In panel (a) we also fix $f_2 = 1.23$, while in panel (b) we fix $f_1 = 2.31$.

middle branch ($\overline{P}_M$). A trajectory converging to the equilibrium is shown, together with the coexistent chaotic attractor. It is known that each attractor must attract one kink point. Here F_1 belongs to the chaotic interval while F_2 converges to the attracting fixed point.

The basin of attraction of the fixed point is given by the interval $[F_2,\ F_2^{-1}]$, where F_2^{-1} is the preimage of the kink point in the middle branch, which consists in the immediate basin, and all its preimages of any rank give the total basin. The complementary set gives the basin of the chaotic attractor. Figures 8b and 8c show two bifurcation diagrams obtained by varying f_2 and with the same values of the other parameters. The only difference lies in the initial conditions used. Even if the two initial conditions are close each other (1.51 and 1.53) there is a large range of values of the parameter f_2 where one trajectory converges to the equilibrium and the other to the chaotic attractor. This causes a sensitivity with respect to the initial levels of price, that is a small perturbation like an external shock may lead to trajectories converging to different attractor (see Bischi and Lamantia (2005) and Bischi *et al.* (2010b) for example). A closer look at Figure 8c tells us that our model is able to reproduce the usual fear and greed scenario, this means that the model is able to explain periods of persistent misalignments and the emergence of endogenous bubbles and crashes. Indeed, some combinations of parameters can produce complex fear and greed market dynamics where we may observe large price drops more/less frequently. The analysis of multistability can be important for regulators and policy makers, as in Tramontana *et al.* (2013), in order to prevent a market from collapsing. Moreover, it gives a first idea of the possible consequences of the introduction of some stochastic elements in the picture, producing a switching from the basin of attractions of a stable equilibrium (a calm scenario) to the basin of attraction of a chaotic attractor

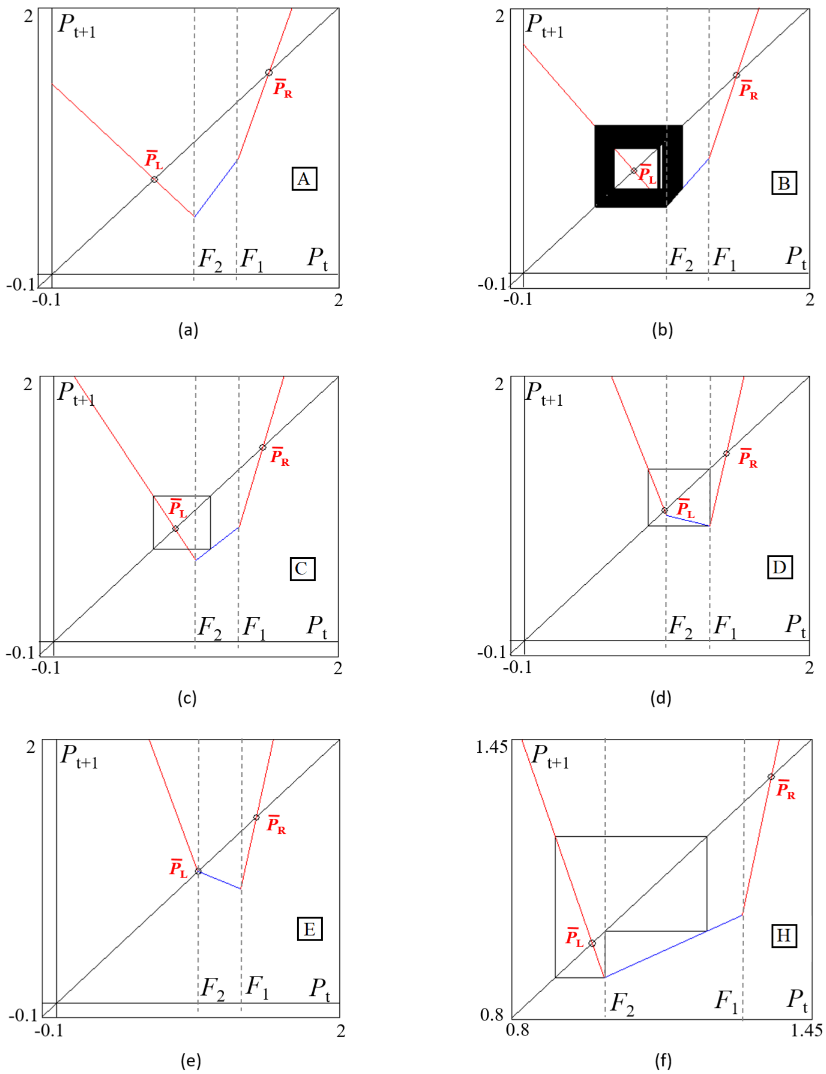

Figure 6: In blue combinations of parameters leading to an equilibrium, in yellow combinations leading to a cycle of period 2, in red to a cycle of period 3. In white chaotic dynamics, while in grey diverging trajectories.

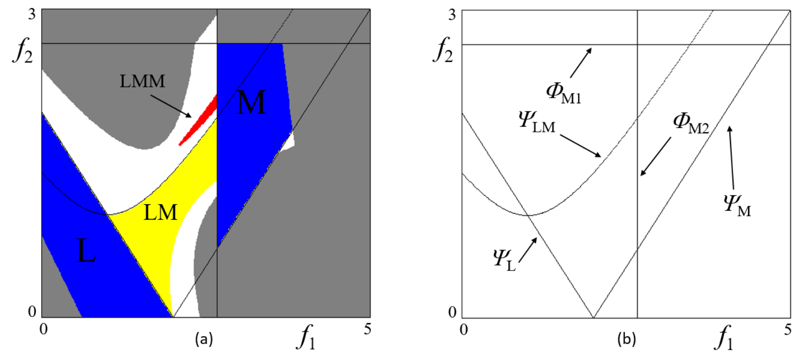

Figure 7: Bifurcation curves obtained keeping fixed the parameters $F_1 = 1.3$, $F_2 = 1$, $m = 0.8$ and by letting vary f_1 and f_2.

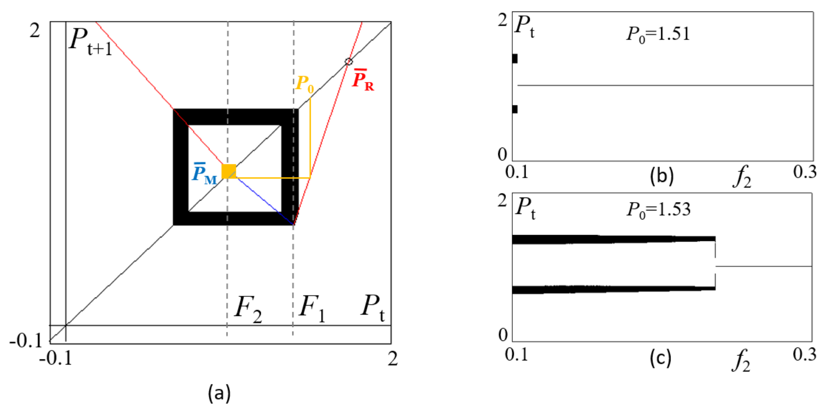

Figure 8: For the three panel we use $m = 0.8$, $F_1 = 1.4$, $F_2 = 1$, $f_1 = 2.05$. In panel (a) $f_2 = 0.15$.

(turbulent scenario), and vice versa.

5 Conclusions

In this paper we have studied a financial market model where interaction comes from two types of fundamentalists that display heterogeneous beliefs in the fundamental value of the asset but homogeneous trading strategies. The model differs from the researches in the field of the PWL maps provided in the last years in the role that agents play in the market. Indeed, considering only fundamentalists, we are assuming a marginal role of speculation. Instead, introducing uncertainty about the fundamental value in the market we are focused on the analysis of market risk inherent to the fundamentals of the market.

For this purpose, our model is able to generate interesting endogenous price dynamics. The final map is piecewise-linear and composed of three linear branches. We have conducted an in-depth analytical study of the model. From the general properties of the map, we conclude on the stability and location of the fixed points. In particular, we prove that the fixed point located in the right branch of the map is always unstable. As a result, depending on the slope of each branch, we are able to analyze three different cases. The first one is characterized by three linear increasing branches and we have seen the occurrence of three kind of scenarios. No equilibria exist and only diverging trajectories may occur.

The second and third case show more interesting dynamics. As we stressed in the paper, with PWL maps the structure of bifurcations is different than those occurring in regular maps. We analytically obtained the most of the bifurcation curves in the f_1, f_2 parameter plane.

The final part of the work concerns the role of multistability. This scenario is characterized by the coexistence of multiple attractors. We shown that depending on the initial condition, the model exhibits a convergence to a quite different attractor, being possible to shift from stable attractor to chaotic attractor. In this case, we are able to replicate several fear and greed market dynamics where we observe more/less price drops.
We have also highlighted the role of the analysis of coexisting attractors for regulators and policy makers. Indeed, tuning the parameters of the model they may prevent dangerous market crashes and guarantee a stability period.

Our model may be extended in several ways. First, it would be interesting to calibrate or even estimate a stochastic version of the model in order to determine how it is able to replicate the stylized facts of real financial markets. Second, we may introduce a speculative component in the model considering chartists and imitators following Brianzoni and Campisi (2020) and Tramontana *et al.* (2014). Finally, one could assume more complex demand functions or that some traders are not always active in the market in line with Tramontana *et al.* (2014).

References

Anufriev, M., Gardini, L., and Radi, D. (2020). Chaos, border collisions and stylized empirical facts in an asset pricing model with heterogeneous agents. *Nonlinear Dynamics*, **102**, 993–1017.

Avrutin, V., Gardini, L., Sushko, I., and Tramontana, F. (2019). *Continuous and Discontinuous Piecewise Smooth One-Dimensional Maps: Invariant Sets and Bifurcation structures*, volume 95. World Scientific.

Bischi, G. I. and Lamantia, F. (2005). Coexisting attractors and complex basins in discrete-time economic models. In *Nonlinear dynamical systems in economics*, pages 187–231. Springer.

Bischi, G.-I., Gallegati, M., Gardini, L., Leombruni, R., and Palestrini, A. (2006). Herd behavior and nonfundamental asset price fluctuations in financial markets. *Macroeconomic Dynamics*, **10**(4), 502.

Bischi, G.-I., Gardini, L., and Tramontana, F. (2010a). Bifurcation curves in discontinuous maps. *Discrete & Continuous Dynamical Systems-B*, **13**(2), 249.

Bischi, G.-I., Chiarella, C., Kopel, M., Szidarovszky, F., *et al.* (2010b). *Nonlinear oligopolies*. Springer.

Brianzoni, S. and Campisi, G. (2020). Dynamical analysis of a financial market with fundamentalists, chartists, and imitators. *Chaos, Solitons & Fractals*, **130**, 109434.

Brianzoni, S., Mammana, C., and Michetti, E. (2010). Updating wealth in an asset pricing model with heterogeneous agents. *Discrete Dynamics in Nature and Society*, **2010**.

Brock, W. A. and Hommes, C. H. (1998). Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control*, **22**(8-9), 1235–1274.

Campisi, G. and Muzzioli, S. (2020). Investor sentiment and trading behavior. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, **30**(9), 093103. doi: 10.1063/5.0011636.

Campisi, G. and Tramontana, F. (2020). Come together: The role of cognitively biased imitators in a small scale agent-based financial market. In *Games and Dynamics in Economics*, pages 69–88. Springer.

Chiarella, C., Dieci, R., and Gardini, L. (2002). Speculative behaviour and complex asset price dynamics: a global analysis. *Journal of Economic Behavior & Organization*, **49**(2), 173–197.

Chiarella, C., Dieci, R., and Gardini, L. (2005). The dynamic interaction of speculation and diversification. *Applied Mathematical Finance*, **12**(1), 17–52.

Chiarella, C., Dieci, R., and Gardini, L. (2006). Asset price and wealth dynamics in a financial market with heterogeneous agents. *Journal of Economic Dynamics and Control*, **30**(9-10), 1755–1786.

Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, **1**, 223–236.

Day, R. H. (1997). Complex dynamics, market mediation and stock price behavior. *North American Actuarial Journal*, **1**(3), 1–16.

Day, R. H. and Huang, W. (1990). Bulls, bears and market sheep. *Journal of Economic Behavior & Organization*, **14**(3), 299–329.

De Grauwe, P. and Kaltwasser, P. R. (2012). Animal spirits in the foreign exchange market. *Journal of Economic Dynamics and Control*, **36**(8), 1176–1192.

Dercole, F. and Radi, D. (2020). Does the "uptick rule" stabilize the stock market? insights from adaptive rational equilibrium dynamics. *Chaos, Solitons & Fractals*, **130**, 109426.

Dieci, R. and Westerhoff, F. (2010). Heterogeneous speculators, endogenous fluctuations and interacting markets: a model of stock prices and exchange rates. *Journal of Economic Dynamics and Control*, **34**(4), 743–764.

Elyasiani, E., Gambarelli, L., and Muzzioli, S. (2020). Moment risk premia and the cross-section of stock returns in the european stock market. *Journal of Banking & Finance*, **111**, 105732.

Franke, R. and Westerhoff, F. (2016). Why a simple herding model may generate the stylized facts of daily returns: explanation and estimation. *Journal of Economic Interaction and Coordination*, **11**(1), 1–34.

Gardini, L. and Tramontana, F. (2011). Border collision bifurcation curves and their classification in a family of 1d discontinuous maps. *Chaos, Solitons & Fractals*, **44**(4-5), 248–259.

Gu, E.-G. (2018). On the existence of chaos in a discontinuous area-preserving map arising in financial markets. *International Journal of Bifurcation and Chaos*, **28**(14), 1850177.

Gu, E.-G. and Guo, J. (2019). Bcb curves and contact bifurcations in piecewise linear discontinuous map arising in a financial market. *International Journal of Bifurcation and Chaos*, **29**(02), 1950022.

He, X.-Z. and Li, Y. (2007). Power-law behaviour, heterogeneity, and trend chasing. *Journal of Economic Dynamics and Control*, **31**(10), 3396–3426.

He, X.-Z. and Zheng, H. (2016). Trading heterogeneity under information uncertainty. *Journal of Economic Behavior & Organization*, **130**, 64–80.

Huang, W. and Day, R. (1993). Chaotically switching bear and bull markets: the derivation of stock price distributions from behavioral rules. *Nonlinear dynamics and evolutionary economics*, pages 169–182.

Huang, W., Zheng, H., and Chia, W.-M. (2010). Financial crises and interacting heterogeneous agents. *Journal of Economic Dynamics and Control*, **34**(6), 1105–1122.

Jungeilges, J., Maklakova, E., and Perevalova, T. (2021). Asset price dynamics in a "bull and bear market". *Structural Change and Economic Dynamics*, **56**, 117–128.

Lux, T. and Marchesi, M. (2000). Volatility clustering in financial markets: a microsimulation of interacting agents. *International journal of Theoretical and Applied Finance*, **3**(04), 675–702.

Naimzada, A. K. and Ricchiuti, G. (2008). Heterogeneous fundamentalists and imitative processes. *Applied Mathematics and Computation*, **199**(1), 171 – 180.

Nusse, H. E. and Yorke, J. A. (1992). Border-collision bifurcations including "period two to period three" for piecewise smooth systems. *Physica D: Nonlinear Phenomena*, **57**(1-2), 39–57.

Nusse, H. E., Ott, E., and Yorke, J. A. (1994). Border-collision bifurcations: An explanation for observed bifurcation phenomena. *Physical Review E*, **49**(2), 1073.

Panchuk, A., Sushko, I., and Westerhoff, F. (2018). A financial market model with two discontinuities: Bifurcation structures in the chaotic domain. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, **28**(5), 055908.

Sushko, I., Tramontana, F., Westerhoff, F., and Avrutin, V. (2015). Symmetry breaking in a bull and bear financial market model. *Chaos, Solitons & Fractals*, **79**, 57–72.

Tramontana, F. and Westerhoff, F. (2013). One-dimensional discontinuous piecewise-linear maps and the dynamics of financial markets. In *Global analysis of dynamic models in economics and finance*, pages 205–227. Springer.

Tramontana, F. and Westerhoff, F. (2016). Piecewise-linear maps and their application to financial markets. *Frontiers in Applied Mathematics and Statistics*, **2**, 10.

Tramontana, F., Gardini, L., Dieci, R., and Westerhoff, F. (2009). The emergence of bull and bear dynamics in a nonlinear model of interacting markets. *Discrete Dynamics in Nature and Society*, **2009**.

Tramontana, F., Gardini, L., and Westerhoff, F. (2010a). Intricate asset price dynamics and one-dimensional discontinuous maps. *Nonlinear Economic Dynamics*, pages 43–57.

Tramontana, F., Westerhoff, F., and Gardini, L. (2010b). On the complicated price dynamics of a simple one-dimensional discontinuous financial market model with heterogeneous interacting traders. *Journal of Economic Behavior & Organization*, **74**(3), 187–205.

Tramontana, F., Westerhoff, F., and Gardini, L. (2013). The bull and bear market model of huang and day: Some extensions and new results. *Journal of Economic Dynamics and Control*, **37**(11), 2351–2370.

Tramontana, F., Westerhoff, F., and Gardini, L. (2014). One-dimensional maps with two discontinuity points and three linear branches: mathematical lessons for understanding the dynamics of financial markets. *Decisions in Economics and Finance*, **37**(1), 27–51.

Tramontana, F., Westerhoff, F., and Gardini, L. (2015). A simple financial market model with chartists and fundamentalists: market entry levels and discontinuities. *Mathematics and Computers in Simulation*, **108**, 16–40.

Westerhoff, F. H. (2004). Greed, fear and stock market dynamics. *Physica A: Statistical Mechanics and its Applications*, **343**, 635–642.